



# The geometry of surfaces contact

J. Siegl<sup>a,\*</sup>, J. Švígler<sup>a</sup>

<sup>a</sup> Faculty of Applied Sciences, UWB in Pilsen, Univerzitní 22, 301 00 Pilsen, Czech Republic

Received 10 September 2007; received in revised form 10 October 2007

## Abstract

This contribution deals with a geometrical exact description of contact between two given surfaces which are defined by the vector functions. These surfaces are substituted at a contact point by approximate surfaces of the second order in accordance with the Taylor series and consequently there is derived a differential surface of these second order surfaces. Knowledge of principal normal curvatures, their directions and the tensor (Dupin) indicatrix of this differential surface are necessary for description of contact of these surfaces. For description of surface geometry the first and the second surface fundamental tensor and a further methods of the differential geometry are used. A geometrical visualisation of obtained results of this analysis is made. Method and results of this study will be applied to contact analysis of tooth screw surfaces of screw machines.

© 2007 University of West Bohemia. All rights reserved.

**Keywords:** contact mechanics, differential geometry, first and second fundamental tensor, Gaussian and mean curvature, principal curvatures and their directions, screw machine, Taylor series, tensor indicatrix

## 1. Introduction

The aim of this paper, which creates the first part of contact analysis of two bodies in accordance with the Hertz theory [2], [3], is the determination of differential surface and its curvatures at this contact point. In this study the simplified surfaces, which represent the complicated technical surface, are considered. The surfaces are given by vector functions. Both surfaces are replaced in the contact point by approximate surfaces of second order in accordance with the Taylor series, [4], pp. 205. At the contact point on this differential surface the principal normal curvatures and their principal directions, which define a *contact base*, are determined. The principle normal curvatures at the contact point must be known in order to describe the contact of surfaces in the manner of the Hertz theory. Therefore the derivatives of this differential surface are necessary up to the second order made. All descriptions are shown for the surface  $\sigma_3$  only. For the surface  $\sigma_2$  is valid the same procedure. Obtained method and results of the solution will be applied for determination of contact of tooth surfaces of screw compressor rotors or screw machine with, as consequence of force and heat deformation of machine housing, skew axes. After that it is possible to deal with the displacements field and stress field in a neighbourhood of the contact point depending on geometry of tooth surfaces.

## 2. Problem description and input parameters

Two screw surfaces  $\sigma_3$  and  $\sigma_2$ , fig. 1, create a general kinematic couple in the space. Their rotation axes are  $o_3$  and  $o_2$ . The initial global coordinate system  $R_1$  given by  $\{O_1; \mathbf{e}_{iR_1}\}$  is placed on the  $o_{30}$  axis, fig. 1. It is considered with respect to a change of its position that the axis

\*Corresponding author. Tel.: +420 377 632 381, e-mail: [jsiegl@kme.zcu.cz](mailto:jsiegl@kme.zcu.cz).

$o_{30}$  is displaced into the position  $o_3$ . This axis displacement is determined by radius vectors  $\mathbf{r}_{A_{30}}, \mathbf{r}_{B_{30}}$  and displacement vectors  $\mathbf{u}_{A_3}, \mathbf{u}_{B_3}$ . The radius vectors are expressed by a homogeneous coordinates. Let these surfaces contact itself at the point  $C \equiv C_2 \equiv C_3$  which are given by the following radius vectors

$${}_{R_{\sigma_3}}\mathbf{r}_{C_3} = [\theta_{C_3} \quad \varphi_{C_3} \quad 1]^T, \quad {}_{R_{\sigma_2}}\mathbf{r}_{C_2} = [\theta_{C_2} \quad \varphi_{C_2} \quad 1]^T, \quad (1)$$

where  $R_{\sigma_i}, i \in \{3, 2\}$  is the surface  $\sigma_i$  coordinate system of the Euclidean affine space  $E_2$  and  $\theta_{C_i}, \varphi_{C_i}$  are curvilinear coordinates of this point  $C_i$  on the surface  $\sigma_i$ . The determination of the contact point of surfaces  $\sigma_3$  and  $\sigma_2$  is not subject of this solution. This problem will be solved separately with the creation of individual surfaces.

For the solution these following parameters are selected. The initial position of the axis  $o_3$ , which is marked as  $o_{30}$ , is coincident with the third base vector  $\mathbf{e}_{3R_1}$ , fig. 1. The position of the axis  $o_3$  determine these following parameters  ${}_{R_1}\mathbf{r}_{A_{30}} = [0 \quad 0 \quad -1 \quad 1]^T$ ,  ${}_{R_1}\mathbf{r}_{B_{30}} = [0 \quad 0 \quad 3 \quad 1]^T$ ,  ${}_{R_1}\mathbf{u}_{A_3} = [1 \quad -1 \quad -1]^T$ ,  ${}_{R_1}\mathbf{u}_{B_3} = [3 \quad -2 \quad 2]^T$ . The surface  $\sigma_3$  is defined with these parameters  $R_{31} = 1$ ;  $R_{3r} = 0,5$ ;  $R_{3z} = 1$ ;  $H_3 = 3$ ;  $n_{R3} = 1,75$  and similary the surface  $\sigma_2$  has these parameters  $R_2 = 1$ ;  $H_2 = 4$ ;  $n_{R2} = 1,75$ . The explanation of these parameters is in the next chapter. For determination of the axis  $o_2$  position are used following two parameters. The first one is  $C_{rotation} = 80 [^\circ]$  and means a rotation around the normal line at the contact point, the second one  $C_{distance}$  has a function for a visual demonstration only which defines the distance between contact points on the contact normal line  $n$ , fig. 1. A turning of the surface  $\sigma_3$  towards the fixed coordinate system  $R_{3f}$ , fig. 1, is given by the coordinate  $\varphi_3 = 120 [^\circ]$ . The contact point  $C_3$  on the surface  $\sigma_3$  is determined by  ${}_{R_{\sigma_3}}C_3 = [\theta_{C_3}, \varphi_{C_3}] = [3\pi/2, 0] \in \Omega_{\sigma_3} \subset \mathbb{R}^2$  and on the surface  $\sigma_2$  by  ${}_{R_{\sigma_2}}C_2 = [\theta_{C_2}, \varphi_{C_2}] = [3,75\pi/2, 0] \in \Omega_{\sigma_2} \subset \mathbb{R}^2$ .

### 3. Geometry definition of problem

The fixed coordinate system  $R_{3f}$  was introduced. This system is determined by the vectors  $\mathbf{r}_{A_{30}}, \mathbf{r}_{B_{30}}, \mathbf{u}_{A_3}, \mathbf{u}_{B_3}$ , fig. 1. Because the surface  $\sigma_3$  has one degree of freedom there is created an actual coordinate system  $R_3$  which coordinate is  $\varphi_3$ . The equation for transformation of the coordinate system  $R_3$  into  $R_1$  is

$${}_{R_1}\mathbf{r} = \mathbf{T}_{R_{3f}, R_1} \mathbf{T}_{R_3, R_{3f}}(\varphi_3) {}_{R_3}\mathbf{r}, \quad (2)$$

where

$$\mathbf{T}_{R_{3f}, R_1} = \begin{bmatrix} {}_{R_1}\mathbf{e}_{1R_{3f}} & {}_{R_1}\mathbf{e}_{2R_{3f}} & {}_{R_1}\mathbf{e}_{3R_{3f}} & {}_{R_1}\mathbf{r}_{O_{3f}} \\ 0 & 0 & 0 & \end{bmatrix}, \quad \mathbf{T}_{R_3, R_{3f}}(\varphi_3) = \begin{bmatrix} \cos \varphi_3 & -\sin \varphi_3 & 0 & 0 \\ \sin \varphi_3 & \cos \varphi_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (3)$$

$\mathbf{T}_{R_b, R_a}$  is the transformation matrix of the vector coordinates in the coordinate system  $R_b$  into the coordinate system  $R_a$ ,  ${}_{R_a}^{\sigma_a}\mathbf{r}_A$  is the radius vector of the point  $A$  on the surface  $\sigma_a$  in the coordinate system  $R_a$  and  ${}_{R_a}\mathbf{e}_{iR_b(j)}$  is the  $j$ -th coordinate of the  $i$ -th base vector of the coordinate system  $R_b$  expressed in the coordinate system  $R_a$ . The displacement of the origin  $O_{30}$  is

$${}_{R_1}\mathbf{r}_{O_{3f}} = {}_{R_1}\mathbf{r}_{A_{30}} + {}_{R_1}\mathbf{u}_{A_3} + \left| {}_{R_1}\mathbf{r}_{A_{30}} \right| {}_{R_1}\mathbf{e}_{3R_{3f}}. \quad (4)$$

Both surfaces  $\sigma_3$  and  $\sigma_2$  are defined in the actual coordinate systems,  $R_3$  and  $R_2$ , by vector functions as follows

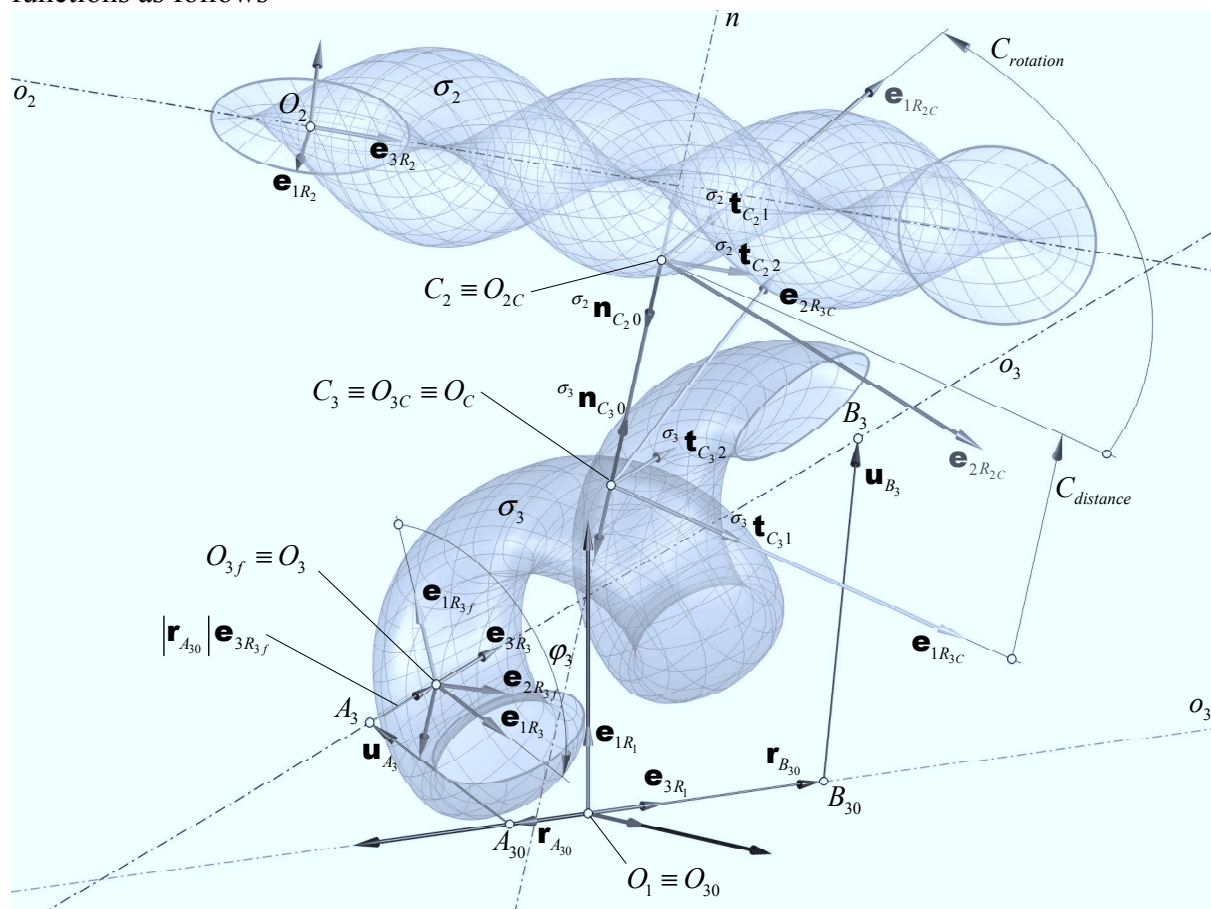


Fig. 1. Visualisation of definite surfaces and coordinate systems.

$$\begin{aligned} \frac{\sigma_3}{R_3} \mathbf{r} = \frac{\sigma_3}{R_3} \mathbf{r}(\theta, \varphi) &= \left[ R_{31} \cos \theta + R_{3r} \cos \varphi \cos \theta, R_{31} \sin \theta + R_{3r} \cos \varphi \sin \theta, R_{3z} \sin \varphi + \frac{H_3}{2\pi} \theta, 1 \right]^T, \\ [\theta, \varphi] &\in \Omega_{\sigma_3} \subset \mathbb{R}^2, \quad \Omega_{\sigma_3} = \langle 0, 2\pi \rangle \times \langle 0, 2\pi n_{R3} \rangle, \end{aligned} \quad (5)$$

where  $R_{31}$ ,  $R_{3r}$ ,  $R_{3z}$  are radiuses of an anuloid,  $H_3$  is a height per one rotate in the direction of the third coordinate,  $n_{R3}$  is a revolution multiplicator and the  $\Omega$  is the 2D range on which the vector function  $\mathbf{r}$  is given, its input. The special cases of the surface described by the (5) are for example ellipsoid, anuloid, circle surface, Corkscrew surface, helicoid, screw anuloid etc. The surface  $\sigma_2$  is defined by

$$\begin{aligned} \frac{\sigma_2}{R_2} \mathbf{r} = \frac{\sigma_2}{R_2} \mathbf{r}(\theta, \varphi) &= \left[ R_2 \cos \varphi \cos \theta, R_2 \cos \varphi \sin \theta, R_2 \sin \varphi + \frac{H_2}{2\pi} \theta, 1 \right]^T, \\ [\theta, \varphi] &\in \Omega_{\sigma_2} \subset \mathbb{R}^2, \quad \Omega_{\sigma_2} = \langle 0, 2\pi \rangle \times \langle 0, 2\pi n_{R_2} \rangle, \end{aligned} \quad (6)$$

where  $R_2$  is a radius of this screw surface,  $H_2$  is a height per one rotate in the direction of the third coordinate,  $n_{R2}$  is a revolution multiplicator. This surface is sometimes called as the Corkscrew surface. The  $\theta$ ,  $\varphi$  are curvilinear coordinates on the surface  $\sigma_i$ . The tangent vectors fields determining a base of curvilinear coordinates in every point of surface  $\sigma_3$  are

$$\sigma_3 \mathbf{t}_1 = \sigma_3 \mathbf{t}_1(\theta, \varphi) = \frac{\partial [\sigma_3 \mathbf{r}(\theta, \varphi)]}{\partial \theta} = \left[ -R_{31} \sin \theta - R_{3r} \cos \varphi \sin \theta, R_{31} \cos \theta + R_{3r} \cos \varphi \cos \theta, \frac{H_3}{2\pi}, 0 \right]^T, \quad (7)$$

$$\sigma_3 \mathbf{t}_2 = \sigma_3 \mathbf{t}_2(\theta, \varphi) = \frac{\partial [\sigma_3 \mathbf{r}(\theta, \varphi)]}{\partial \varphi} = [-R_{3r} \sin \varphi \cos \theta, -R_{3r} \sin \varphi \sin \theta, R_{3z} \cos \varphi, 0]^T. \quad (8)$$

The normal and the unit normal vector is

$$\sigma_3 \mathbf{n}(\theta, \varphi) = \sigma_3 \mathbf{t}_1 \times \sigma_3 \mathbf{t}_2, \quad \sigma_3 \mathbf{n}_0(\theta, \varphi) = \sigma_3 \mathbf{n} \left| \sigma_3 \mathbf{n} \right|^{-1}. \quad (9)$$

On the surface  $\sigma_3$  is selected arbitrary point  $C_3$ , its radius vector in the  $R_3$  is  $\sigma_3 \mathbf{r}_{C_3} = \sigma_3 \mathbf{r}_{C_3}(\theta_{C_3}, \varphi_{C_3})$ , which determines the contact point. At this point is established a coordinate system  $R_{3C} \equiv \{O_{3C}; \mathbf{e}_{iR_{3C}}\}$ , the transformation matrix of this system into  $R_3$  is

$$\mathbf{T}_{R_{3C}, R_3} = \begin{bmatrix} R_3 \mathbf{e}_{1R_{3C}} & R_3 \mathbf{e}_{2R_{3C}} & R_3 \mathbf{e}_{3R_{3C}} & \sigma_3 \mathbf{r}_{C_3} \\ 0 & 0 & 0 & \end{bmatrix} = \begin{bmatrix} \sigma_3 \mathbf{t}_{C_3 1} \left| \sigma_3 \mathbf{t}_{C_3 1} \right|^{-1} & \sigma_3 \mathbf{n}_{C_3 0} \times \sigma_3 \mathbf{t}_{C_3 1} \left| \sigma_3 \mathbf{t}_{C_3 1} \right|^{-1} & \sigma_3 \mathbf{n}_{C_3 0} & \sigma_3 \mathbf{r}_{C_3} \\ 0 & 0 & 0 & \end{bmatrix}. \quad (10)$$

A similar coordinate system  $R_{2C} \equiv \{O_{2C}; \mathbf{e}_{iR_{2C}}\}$  is created for the surface  $\sigma_2$  in which origin will be to lie the contact point  $C_2$  of the surface  $\sigma_2$ . This system is defined with the transformation matrix  $\mathbf{T}_{R_{2C}, R_3}(C_{distance}, C_{rotation}, \pi)$ , fig. 1. On the surface  $\sigma_2$  is determined arbitrary point  $C_2$ , its radius vector in the  $R_2$  is  $\sigma_2 \mathbf{r}_{C_2} = \sigma_2 \mathbf{r}_{C_2}(\theta_{C_2}, \varphi_{C_2})$ , which determines the contact point on the surface  $\sigma_2$ . At this point is determined a general coordinate system on the surface  $\sigma_2$   $\{C_2; \sigma_2 \mathbf{t}_{C_2 1}, \sigma_2 \mathbf{t}_{C_2 2}, \sigma_2 \mathbf{n}_{C_2 0}\}$ , which the first base  $\sigma_2 \mathbf{t}_{C_2 1}$  is colinear with the  $\mathbf{e}_{1R_{2C}}$  and the third base  $\sigma_2 \mathbf{n}_{C_2 0}$  is colinear with the  $\mathbf{e}_{3R_{2C}}$ , fig. 1. The transformation matrix of the coordinate system  $R_{2C}$  into the coordinate system  $R_2$  has thus the form

$$\mathbf{T}_{R_{2C}, R_2} = \begin{bmatrix} R_2 \mathbf{e}_{1R_{2C}} & R_2 \mathbf{e}_{2R_{2C}} & R_2 \mathbf{e}_{3R_{2C}} & \sigma_2 \mathbf{r}_{C_2} \\ 0 & 0 & 0 & \end{bmatrix} = \begin{bmatrix} \sigma_2 \mathbf{t}_{C_2 1} \left| \sigma_2 \mathbf{t}_{C_2 1} \right|^{-1} & \sigma_2 \mathbf{n}_{C_2 0} \times \sigma_2 \mathbf{t}_{C_2 1} \left| \sigma_2 \mathbf{t}_{C_2 1} \right|^{-1} & \sigma_2 \mathbf{n}_{C_2 0} & \sigma_2 \mathbf{r}_{C_2} \\ 0 & 0 & 0 & \end{bmatrix}, \quad (11)$$

the inverse matrix  $\mathbf{T}_{R_2, R_{2C}} = \mathbf{T}_{R_{2C}, R_2}^{-1}$ . The contact points  $C_3$  and  $C_2$  are identical by the parameter  $C_{distance} = 0$  and therefore they create the contact point  $C$ .

#### 4. The approximation of surfaces with second order surfaces at the contact points

The surface  $\sigma_i$ ,  $i \in \{3, 2\}$ , is substituted at the point  ${}_{R_{\sigma_i}} C_i = [\theta_{C_i} \quad \varphi_{C_i}]$  by the *Taylor series* of the vector function defining the surface  $\sigma_i$  up to the second order, fig. 2, 3. As an illustration the surface  $\sigma_3$  at the contact point  ${}_{R_{\sigma_3}} C_3$  is thus substituted with following approximate surface

$$\begin{aligned} \sigma_3^T \mathbf{r}(\theta, \varphi) &= \sigma_3 \mathbf{r}(\theta_{C_3}, \varphi_{C_3}) + \frac{\partial \sigma_3 \mathbf{r}(\theta_{C_3}, \varphi_{C_3})}{\partial \theta} (\theta - \theta_{C_3}) + \frac{\partial \sigma_3 \mathbf{r}(\theta_{C_3}, \varphi_{C_3})}{\partial \varphi} (\varphi - \varphi_{C_3}) + \\ &+ \frac{1}{2} \left[ \frac{\partial^2 \sigma_3 \mathbf{r}(\theta_{C_3}, \varphi_{C_3})}{\partial \theta \partial \theta} (\theta - \theta_{C_3})^2 + 2 \frac{\partial^2 \sigma_3 \mathbf{r}(\theta_{C_3}, \varphi_{C_3})}{\partial \theta \partial \varphi} (\theta - \theta_{C_3})(\varphi - \varphi_{C_3}) + \frac{\partial^2 \sigma_3 \mathbf{r}(\theta_{C_3}, \varphi_{C_3})}{\partial \varphi \partial \varphi} (\varphi - \varphi_{C_3})^2 \right]. \end{aligned} \quad (12)$$

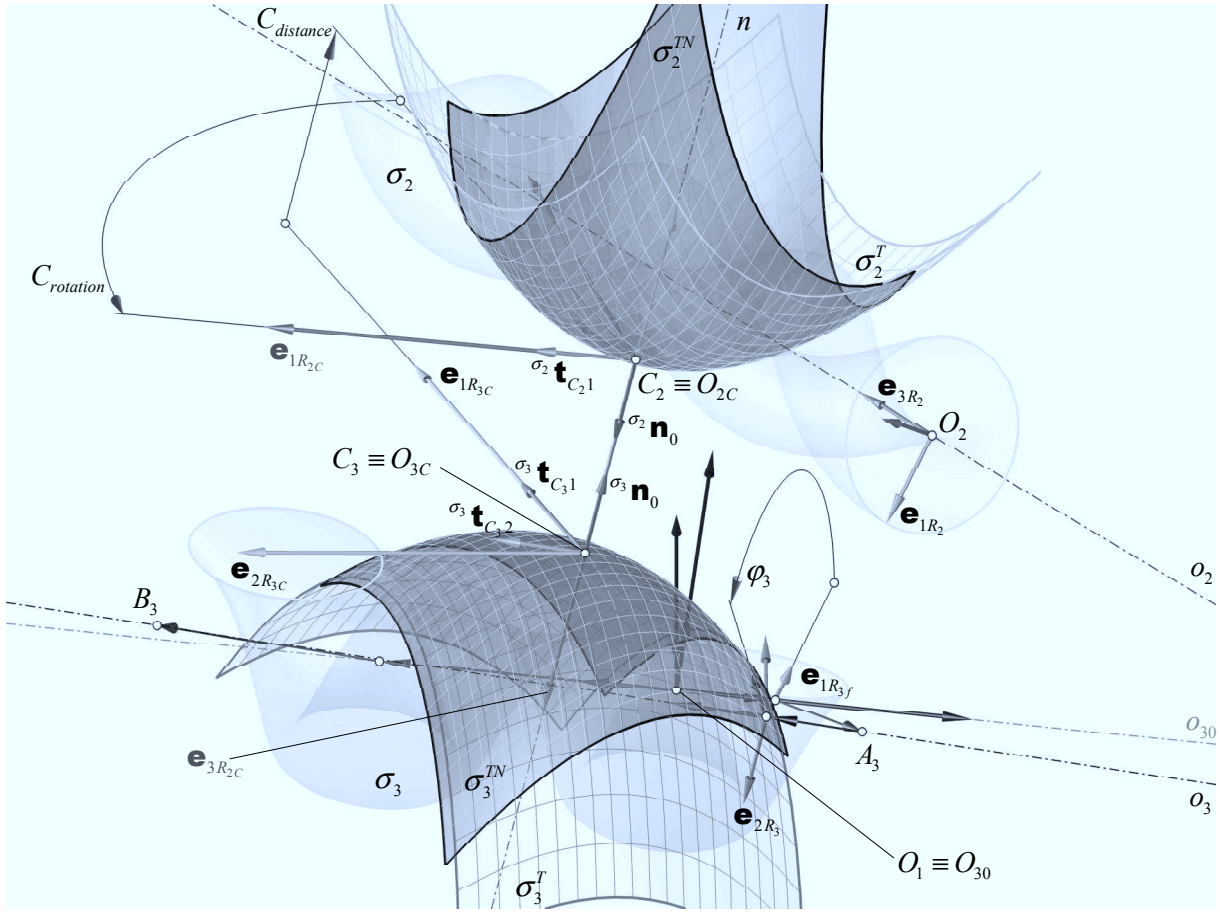


Fig. 2. Approximate surfaces with remeshed approximate surfaces.

This approximate surface  $\sigma_3^T$  is expressed in the coordinate system  $R_{3C}$  with the equation

$$\sigma_{R_{3C}}^T \mathbf{r} = \mathbf{T}_{R_{3C}, R_3}^{-1} \left( \sigma_{R_3}^T \mathbf{r} - \sigma_{R_3}^T \mathbf{r}_{C_3} \right). \quad (13)$$

The approximate surface  $\sigma_2^T$  is expressed in the coordinate system  $R_{3C}$  likewise

$$\sigma_{R_{3C}}^T \mathbf{r} = \mathbf{T}_{R_{2C}, R_{3C}} \mathbf{T}_{R_{2C}, R_2}^{-1} \left( \sigma_{R_2}^T \mathbf{r} - \sigma_{R_2}^T \mathbf{r}_{C_2} \right). \quad (14)$$

For determination of the differential surface  $\sigma_D$  at the contact point  $C$  the first and the second coordinate of these approximate surfaces  $\sigma_3^T$  and  $\sigma_2^T$  have to be selected on an orthonormal plain grid in the coordinate system  $R_{3C}$ , fig. 3. This transformation is given with the system of three nonlinear equations

$$\sigma_{R_{3C}}^{TN} \mathbf{r} = \mathbf{T}_{R_{3C}, R_3}^{-1} \left( \sigma_{R_3}^T \mathbf{r} - \sigma_{R_3}^T \mathbf{r}_{C_3} \right), \quad (15)$$

where coordinates  $\left\{ \sigma_{R_{3C}}^{TN} \mathbf{r}_{(1)}, \sigma_{R_{3C}}^{TN} \mathbf{r}_{(2)} \right\} \in \langle -M, M \rangle$ ,  $M$  is a boundary of the discrete interval,  $N$  index indicates the new surface. The (15) is rewritten into the form  $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ , so

$$\mathbf{T}_{R_{3C}, R_{3C}} \left( \sigma_{R_3}^T \mathbf{r} - \sigma_{R_3}^T \mathbf{r}_{C_3} \right) - \sigma_{R_{3C}}^{TN} \mathbf{r} = \mathbf{0}, \quad (16)$$

where the unknowns vector  $\mathbf{x} = \left[ \begin{matrix} \theta_{R_{\sigma_3^{TN}}} & \varphi_{R_{\sigma_3^{TN}}} & \sigma_{R_{3C}}^{TN} \mathbf{r}_{(3)} \end{matrix} \right]^T$ . For the solution of this equations sys-



tem the *Newton's method* is used. This new approximate surface  $\sigma_i^{TN}$ ,  $i \in \{3, 2\}$ , fig. 4, is actually given in the form  $\sigma_{R_{3C}}^{TN} \mathbf{r} = [u^1, u^2, f_i(u^1, u^2)]^T$ ,  $[u^1, u^2] \in \Omega_{\sigma_i^{TN}}$ , where  $\Omega_{\sigma_i^{TN}}$  is the two-dimensional discrete region and  $f_i$  is a function of two variables on the  $\Omega_{\sigma_i^{TN}}$ . The surface  $\sigma_i^{TN}$  is thus identical with the  $\sigma_i^T$ .

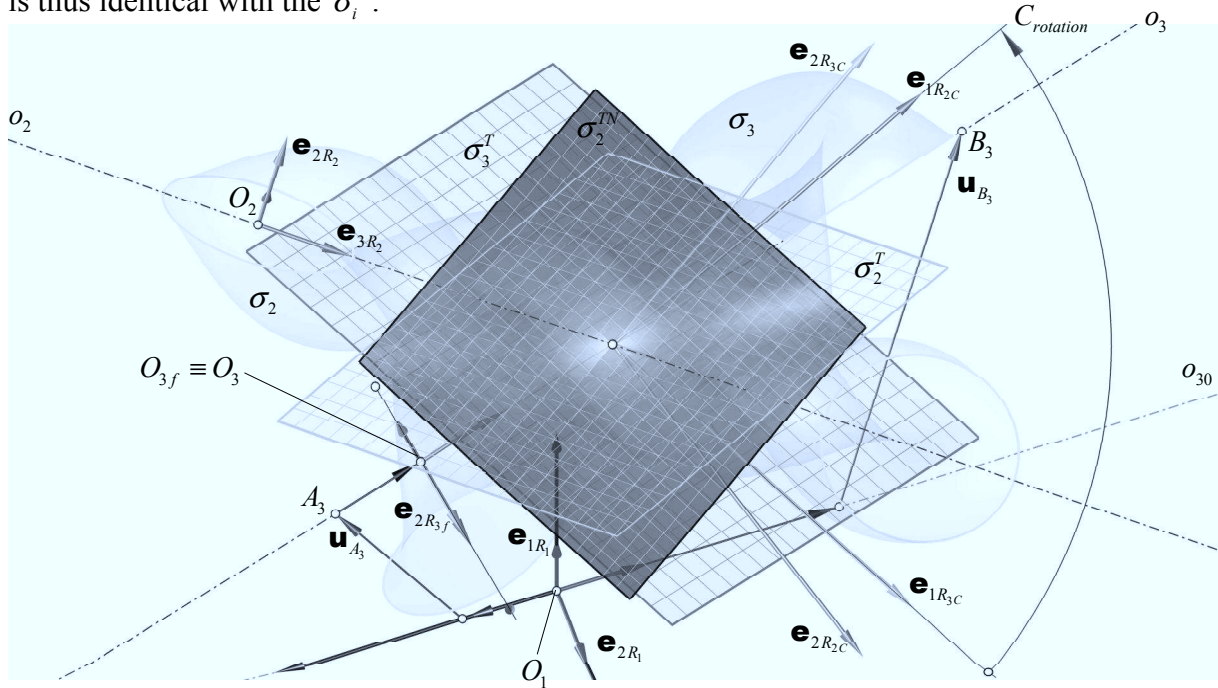


Fig. 3. View of approximate surfaces with remeshed approximate surfaces along the normal line  $n$ .

## 5. Differential surface and inner geometry at the contact point

The differential surface  $\sigma_D$  is described in the coordinate system  $R_{3C}$  with the equation

$$\sigma_D \mathbf{r} = \sigma_D \mathbf{r}(u^1, u^2) = \left( \sum_{i=1}^2 \sigma_{R_{3C}}^{TN} \mathbf{r}_{(i)} \mathbf{e}_{iR_{3C}} \right) + \left( \sigma_{R_{3C}}^{TN} \mathbf{r}_{(3)} - \sigma_{R_{3C}}^{TN} \mathbf{r}_{(3)} \right) \mathbf{e}_{3R_{3C}} + \mathbf{e}_{4R_{3C}}. \quad (17)$$

The differentiations have to be performed numerically, thus differentiation  $\sigma_D \mathbf{r}(u^1, u^2)$  with respect to  $u^i$ ,  $i \in \{1, 2\}$  gives two tangent vector fields determining the base of the local curvilinear coordinates and four vector fields given with the second derivatives

$$\sigma_D \mathbf{t}_i(u^1, u^2) = \frac{\partial \sigma_D \mathbf{r}(u^1, u^2)}{\partial u^i}, \quad \sigma_D \mathbf{r}_{ij}(u^1, u^2) = \frac{\partial \sigma_D \mathbf{r}(u^1, u^2)}{\partial u^i \partial u^j}, \quad i, j \in \{1, 2\}. \quad (18)$$

The normal and the unit normal vector is

$$\sigma_D \mathbf{n}(u^1, u^2) = \sigma_D \mathbf{t}_1 \times \sigma_D \mathbf{t}_2, \quad \sigma_D \mathbf{n}_0(u^1, u^2) = \frac{\sigma_D \mathbf{n}}{|\sigma_D \mathbf{n}|}. \quad (19)$$

The covariant coordinates of the *first fundamental tensor*  $g_{ij}$ , [1], pp. 186, on the surface  $\sigma_D$  at the contact point  ${}_{R_{\sigma_D}} C = [u_C^1, u_C^2] = [0, 0] \in \Omega_{\sigma_D} \subset \mathbb{R}^2$  are defined by the dot product of the tangent vectors  $\sigma_D \mathbf{t}_{Ci}$

$$\mathbf{G} = [g_{ij}] = \frac{\sigma_D}{R_{3C}} \mathbf{t}_{Ci} \cdot \frac{\sigma_D}{R_{3C}} \mathbf{t}_{Cj}, \quad i, j \in \{1, 2\}. \quad (20)$$

The covariant coordinates of the *second fundamental tensor*  $h_{ij}$ , [1], pp. 199, on the surface  $\sigma_D$  at the contact point  $C$  are defined by

$$\mathbf{H} = [h_{ij}] = \frac{\sigma_D}{R_{3C}} \mathbf{n}_{C0} \cdot \frac{\sigma_D}{R_{3C}} \mathbf{r}_{Cij}, \quad i, j \in \{1, 2\}. \quad (21)$$

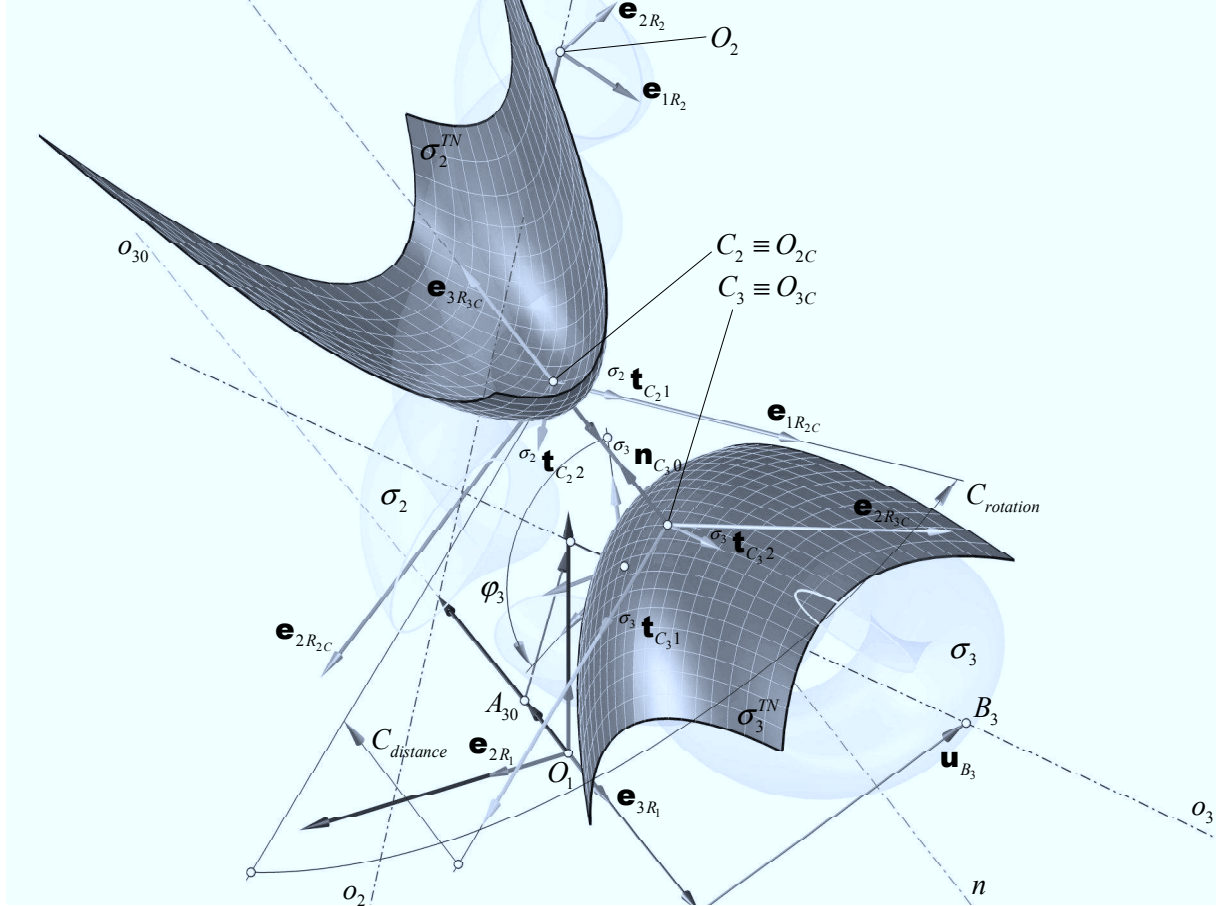


Fig. 4. The remeshed approximate surfaces.

The Gaussian curvature  $K$  and the mean curvature  $H$  of this surface at the contact point  $C$  is given, [1], pp. 215, by

$$K = \frac{\det(\mathbf{H})}{\det(\mathbf{G})} = \frac{h_{11}h_{22} - (h_{12})^2}{g_{11}g_{22} - (g_{12})^2}, \quad H = \frac{1}{2} \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g_{11}g_{22} - (g_{12})^2}. \quad (22)$$

The principal normal curvatures  $\kappa_{1,2}$  are determined from the following equations system

$$K = \kappa_1 \kappa_2, \quad H = \frac{1}{2}(\kappa_1 + \kappa_2) \Rightarrow \kappa_{1,2} = H \pm \sqrt{H^2 - K}. \quad (23)$$

Determination of the principal curvatures and the directions of their normal planes leads up to the generalized problem of eigen values which is described, [6], pp. 288, with the equation

$$\mathbf{H}\mathbf{x} = \lambda\mathbf{G}\mathbf{x}. \quad (24)$$

The solution of this equation gives the eigen values  $\lambda_i$ , that are principal normal curvatures or extrem curvatures actually and eigen vectors  $\mathbf{v}_i$ . These eigen values and vectors are writed

down as follows

$$\mathbf{D} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}, \quad \mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2]. \quad (25)$$

The local coordinate system at a point on a surface is generally an affine coordinate system. In this case at the contact point  $C$  there is the local coordinate system  $R_{DCg} \equiv \{C; \frac{\sigma_D}{R_{3C}} \mathbf{t}_{C1}, \frac{\sigma_D}{R_{3C}} \mathbf{t}_{C2}, \frac{\sigma_D}{R_{3C}} \mathbf{n}_{C0}\}$  orthogonal. The eigen vectors expressed in the coordinate system  $R_{3C}$  are

$${}_{R_{3C}} \mathbf{V} = \mathbf{T}_{R_{DCg}, R_{3C}} \begin{bmatrix} {}_{R_{DCg}} \mathbf{v}_1 & {}_{R_{DCg}} \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{\sigma_D}{R_{3C}} \mathbf{t}_{C1} & \frac{\sigma_D}{R_{3C}} \mathbf{t}_{C2} \end{bmatrix} \begin{bmatrix} {}_{R_{DCg}} \mathbf{v}_1 & {}_{R_{DCg}} \mathbf{v}_2 \end{bmatrix} \quad (26)$$

and the angle between eigen vectors  $\mathbf{v}_i$  is at each point  $[u^1, u^2]$  of  $\sigma_D$

$$\varphi_{\mathbf{v}_1 \mathbf{v}_2} = \varphi_{\mathbf{v}_1 \mathbf{v}_2}(u^1, u^2) = \arccos \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \|\mathbf{v}_2\|} = \frac{\pi}{2}. \quad (27)$$

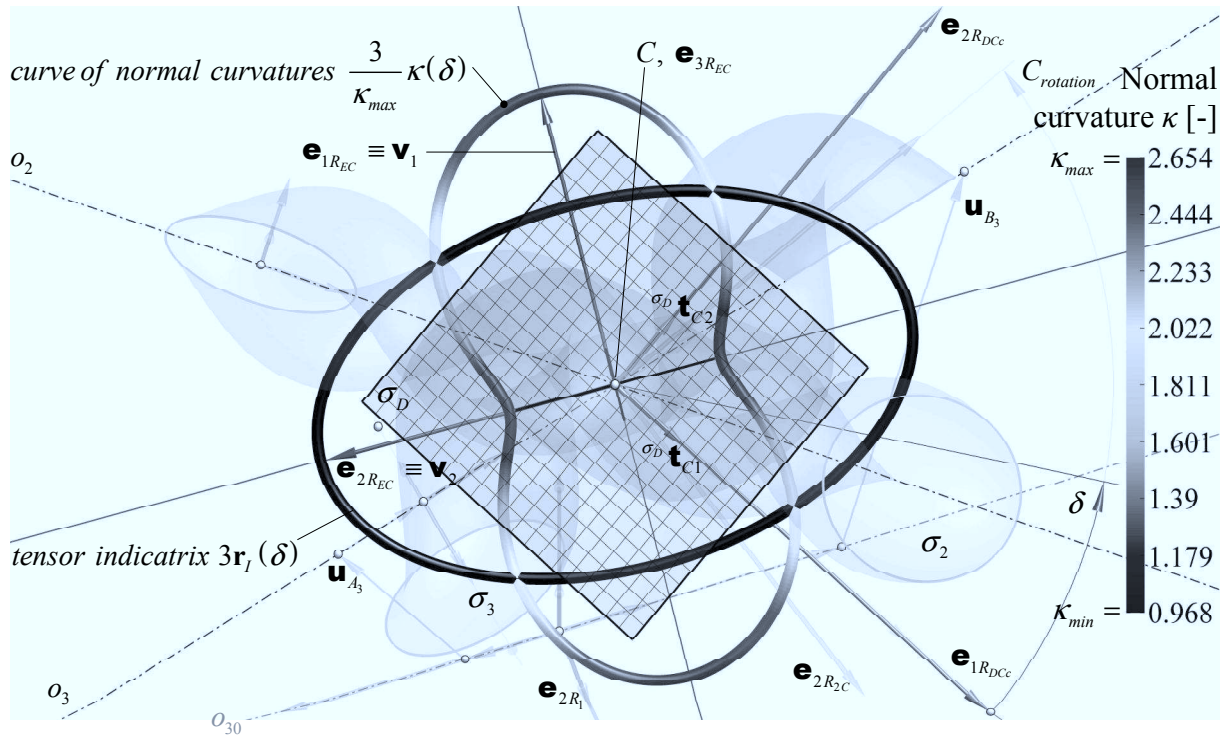


Fig. 5. View along the normal line on tensor indicatrix, curve of normal curvatures and their directions of differential surface at contact point  $C$ .

These unit eigen vectors define the new orthonormal coordinate system  $R_{EC}$  (extreme curvatures) at the contact point  $C$  given by  $\{C; \mathbf{e}_{iR_{EC}}\}$  that it is called *the contact base* as well. The third base vector is defined with

$${}_{R_{3C}} \mathbf{e}_{3R_{EC}} = {}_{R_{3C}} \mathbf{v}_1 \times {}_{R_{3C}} \mathbf{v}_2. \quad (28)$$

For the next step it is necessary at the point  $C$  to create an auxiliary orthonormal coordinate system  $R_{DCc} \equiv \{C; \mathbf{e}_{iDCc}\}$ . The transformation matrix of this system into  $R_{3C}$  is



$$\mathbf{T}_{R_{DCc}, R_{3C}} = \begin{bmatrix} R_{3C} \mathbf{e}_{1DCc} & R_{3C} \mathbf{e}_{2DCc} & R_{3C} \mathbf{e}_{3DCc} & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left[ \begin{array}{c|c|c|c} \frac{\sigma_D \mathbf{t}_{C1}}{R_{3C}} \left| \frac{\sigma_D \mathbf{t}_{C1}}{R_{3C}} \right|^{-1} & \frac{\sigma_D \mathbf{n}_{C0}}{R_{3C}} \times \frac{\sigma_D \mathbf{t}_{C1}}{R_{3C}} \left| \frac{\sigma_D \mathbf{t}_{C1}}{R_{3C}} \right|^{-1} & \frac{\sigma_D \mathbf{n}_{C0}}{R_{3C}} & \mathbf{0} \\ \hline 0 & 0 & 0 & 1 \end{array} \right], \quad (29)$$

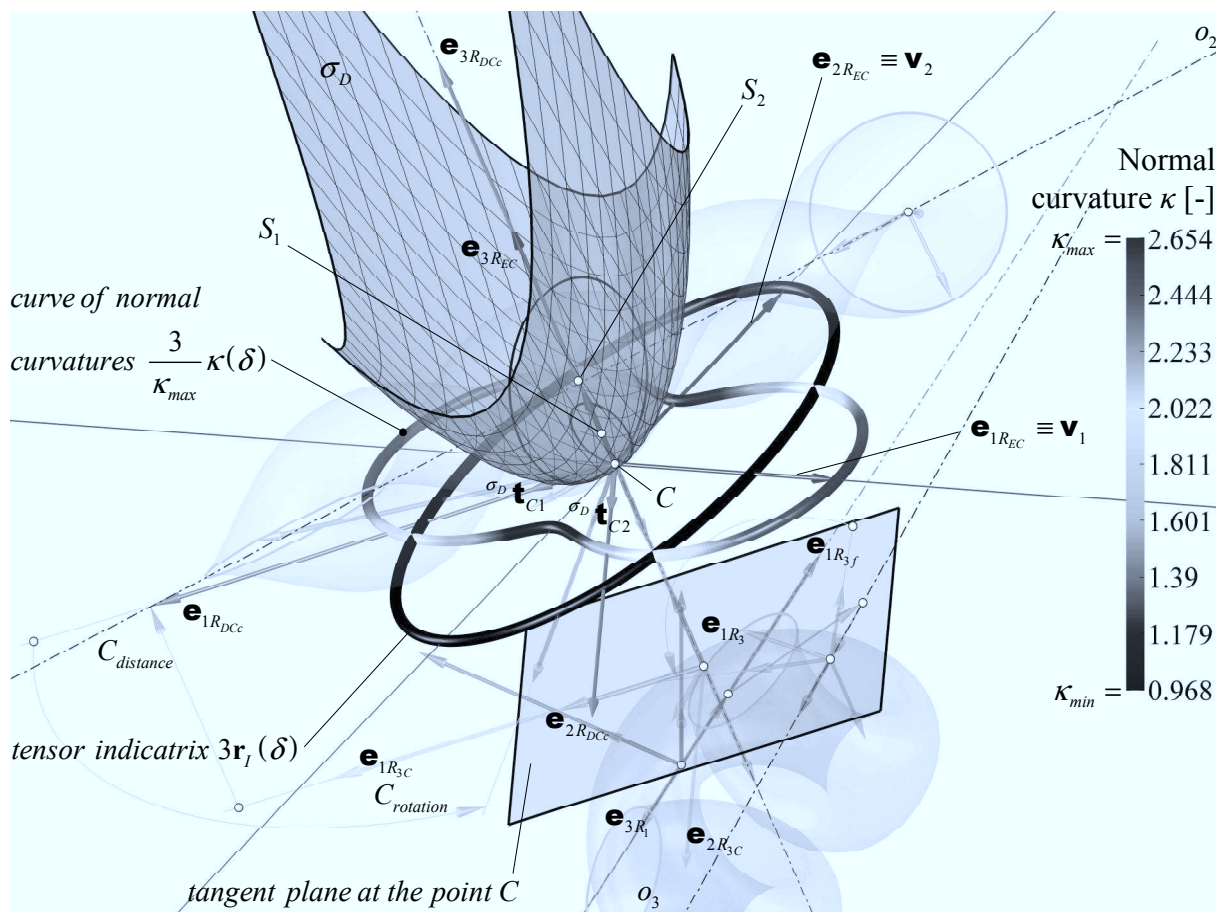


Fig. 6. Tensor indicatrix, curve of normal curvatures and osculate circles in the principle directions, the point  $S_i$  is the circle center.

where  $\mathbf{o} = [0 \ 0 \ 0]^T$ . The affine transformation of an orthonormal coordinate system  $R_{DCc}$  into an affine coordinate system  $R_{DCg}$  is given by

$${}_{R_{DCg}}\mathbf{r} = \mathbf{T}_{R_{DCc}, R_{DCg}}(\varphi) {}_{R_{DCc}}\mathbf{r} = \begin{bmatrix} |\sigma_D \mathbf{t}_{C1}|^{-1} & 0 \\ 0 & |\sigma_D \mathbf{t}_{C2}|^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1/\tan(\varphi) \\ 0 & 1/\sin(\varphi) \end{bmatrix} \cdot \begin{bmatrix} {}_{R_{DCc}}x^1 \\ {}_{R_{DCc}}x^2 \end{bmatrix}, \quad (30)$$

where  $\varphi$  is the angle between  ${}^{\sigma_D}\mathbf{t}_{Ci}$  vectors and  $x^i$  is a vector coordinate. Because the base vectors  ${}^{\sigma_D}\mathbf{t}_{Ci}$  are orthogonal in the solved case the angle  $\varphi = \pi/2$ . The relation for the normal curvature  $\kappa(u^1, u^2, \delta)$ , [1], pp. 207, in the normal plane given by vectors  ${}^{\sigma_D}\mathbf{n}_{C0}$  and  $\mathbf{t}(\delta)$  at the contact point  $C$  is

$$\kappa = \kappa(u^1, u^2, \delta) = \frac{h_{ij} t^i t^j}{g_{ij} t^i t^j} = \frac{\begin{pmatrix} R_{DCg} \mathbf{t} \end{pmatrix}^T \mathbf{H} \begin{pmatrix} R_{DCg} \mathbf{t} \end{pmatrix}}{\begin{pmatrix} R_{DCg} \mathbf{t} \end{pmatrix}^T \mathbf{G} \begin{pmatrix} R_{DCg} \mathbf{t} \end{pmatrix}}, \quad (31)$$

where

$${}_{R_{DCg}} \mathbf{t}(\delta) = \mathbf{T}_{R_{DCc}, R_{DCg}}(\varphi) {}_{R_{DCc}} \mathbf{t}(\delta), \quad {}_{R_{DCc}} \mathbf{t}(\delta) = \begin{bmatrix} \cos \delta \\ \sin \delta \end{bmatrix}, \quad \delta \in \langle 0, 2\pi \rangle, \quad i, j \in \{1, 2\}, \quad (32)$$

$\mathbf{t}(\delta)$  is an unit vector in the tangential plane. The equation of the *tensor indicatrix*, i.e. Dupin indicatrix, [1], pp. 209, is

$$|h_{ij} r^i r^j| = 1 = \left| \left( {}_{R_{DCg}} \mathbf{r} \right)^T \mathbf{H} \left( {}_{R_{DCg}} \mathbf{r} \right) \right|, \quad (33)$$

where  $r^i$  is a point coordinate on the indicatrix curve, fig. 5 and 6. In these pictures there is the curve of the tensor indicatrix illustrated on a scale 3 and the curve of normal curvatures on a scale  $3/\kappa_{max}$ . The parametric expression of the indicatrix curve at the point  $C$  can be

$${}_{R_{3C}} \mathbf{r}_I = {}_{R_{3C}} \mathbf{r}_I(\delta) = \frac{\sigma_D}{R_{3C}} \mathbf{r}_{C_2} + \frac{1}{\sqrt{|\kappa(\delta)|}} \left[ {}_{R_{DCg}} t^i(\delta) \right] \left[ \frac{\sigma_D}{R_{3C}} \mathbf{t}_{Ci} \right], \quad \delta \in \langle 0, 2\pi \rangle, \quad i \in \{1, 2\}, \quad (34)$$

where  $t^i(\delta)$  is a coordinate of the unit vector  $\mathbf{t}(\delta)$ .

## 6. Conclusion

This work, which occupies by the geometry of surfaces and their differential surface at the contact point, is the preliminary part of the contact analysis of two surfaces based on the Hertz theory. The aim of this presented analysis is the determination of the differential surface of both surfaces and its curvatures at the contact point. Consequently the contact base, that creates coordinate system, is determined. This theoretical study of the contact geometry will be implemented to the contact of tooth surfaces of screw machines in operation mode when the axes of tooth surfaces are skew. In this case the original contact curve between tooth surfaces changes into the point contact, which causes an increase of the value of normal force at this point of more than eighty times, [5], with respect to normal force at general point of contact curve in case of non-deformed, parallel, position of rotors. This effect can be the cause of a damage of tooth surfaces. The differential surface  $\sigma_D$ , which describes the relative distance of approximate surfaces  $\sigma_3^T$  and  $\sigma_2^T$ , in the neighbourhood of the contact point, is fundamental to the solution of the contact analysis. The next step of this work will be the determination of the displacement field and stress field in the neighbourhood of the contact point of tooth screw surfaces of screw machines.

## Acknowledgement

The work has been supported by the project MSM 4977751303 of the Ministry of Education of the Czech Republic.

## References

- [1] B. Budinský, B. Kepr, Základy diferenciální geometrie s technickými aplikacemi, SNTL, 1970.
- [2] H. Hertz, Über die Berührung fester elastischen Körper, Journal für reine und angewandte Mathematik 92, 1881, pp. 156–171.
- [3] H. Hertz, Über die Verteilung der Druckkräfte in einem elastischen Kreiscylinder. Schlömilch's Zeitschrift für Mathematic und Physic, Volume 28, 1883, pp. 125–128.
- [4] J. Jaeger, New Solutions in Contact Mechanics, WIT Press, Great Britain, 2005.
- [5] J. Švígler, V. Machulda, J. Siegl, Deformation of Screw Machine Housing under Force Field and its Consequences, Schraubenmaschinen 2006, VDI-Berichte 1932, VDI Verlag GmbH, Düsseldorf, 2006, pp. 377–388.
- [6] E. Vitásek, Technický průvodce 67 – Numerické metody, SNTL, Praha, 1980.